Problem Sheet 3

Exercise 3.1.

Let $(\mathcal{X}_i)$ be a collection of separable metric spaces. Prove that the infinite product $\Pi_{i=1}^{\infty}\mathcal{X}_i$, equipped with product topology, is separable.

Exercise 3.2.

Let (x_n) be a sequence in the metric space $\mathcal{X}$, and $x \in \mathcal{X}$. Prove that (x_n) converges to x in $\mathcal{X}$ if and only if the associated sequence of delta measures (δ_{x_n}) converges weakly to δ_x.

Exercise 3.3.

Let $\mathcal{X}$ be a separable metric space, and (μ_n) be a collection of (uniformly) tight probability measures on $\mathcal{X}$.

1. Show that there exists a subsequence (μ_{n_k}) and a probability measure μ on $\mathcal{X}$ such that (μ_{n_k}) is weakly convergent to μ.
2. Let $\varepsilon > 0$ and K be a compact set such that $\mu_n(\mathcal{X}\backslash K) < \varepsilon$ for all n. Prove that $\mu(\mathcal{X}\backslash K) \leq \varepsilon$.

Exercise 3.4.

Let $\mathcal{X}$ be a metric space and $\mathbb{P}(\mathcal{X})$ the space of probability measures on $\mathcal{X}$. Prove that limits in the topology of weak convergence on $\mathcal{X}$ are unique.

Exercise 3.5.

Let $\mathcal{X}$, $\mathcal{Y}$ be metric spaces and (μ_n) a sequence in $\mathbb{P}(\mathcal{X})$, (ν_n) a sequence in $\mathbb{P}(\mathcal{Y})$. Prove that the sequence of product measures $(\mu_n \otimes \nu_n)$ is tight in $\mathbb{P}(\mathcal{X} \times \mathcal{Y})$ if and only if (μ_n) is tight in $\mathbb{P}(\mathcal{X})$ and (ν_n) is tight in $\mathbb{P}(\mathcal{Y})$.

Exercise 3.6.

Let $(\mu_n), \mu \in \mathbb{P}(\mathbb{R}^d)$ satisfy that for all continuous and compactly supported $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} f d\mu_n \longrightarrow \int_{\mathbb{R}^d} f d\mu \tag{1}$$

as $n \to \infty$. Assume in addition that the sequence (μ_n) is tight. Prove that (μ_n) is weakly convergent to μ.

Exercise 3.7.

Let $\mathcal{X}, \mathcal{Y}$ be complete and separable metric spaces, (μ_n) a sequence in $\mathbb{P}(\mathcal{X})$ weakly convergent to some $\mu \in \mathbb{P}(\mathcal{X})$ and (ν_n) a sequence in $\mathbb{P}(\mathcal{Y})$ weakly convergent to some $\nu \in \mathbb{P}(\mathcal{Y})$. Prove that the sequence of product measures $(\mu_n \otimes \nu_n)$ is weakly convergent to $\mu \otimes \nu$ in $\mathbb{P}(\mathcal{X} \times \mathcal{Y})$.